Non-Riemannian geometrical asymmetrical damping stresses on the Lagrange instability of shear flows

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

It is shown that the physical interpretation of Elie Cartan three-dimensional space torsion as couple asymmetric stress, has the effect of damping, previously Riemannian unstable Couette planar shear flow, leading to stability of the flow in the Lagrangean sense. Actually, since the flow speed is inversely proportional to torsion, it has the effect of causing a damping in the planar flow atenuating the instability effect. In this sense we may say that Cartan torsion induces shear viscous asymmetric stresses in the fluid, which are able to damp the instability of the flow. The stability of the flow is computed from the sectional curvature in non-Riemannian three-dimensional manifold. Marginal stability is assumed by making the sectional non-Riemannian curvature zero, which allows us to determine the speeds of flows able to induce this stability. The ideas discussed here show that torsion plays the geometrical role of magnetic field in hydromagnetic instability of Couette flows recently investigated by Bonnano and Urpin (PRE, (2007,in press) can be extended and applied to plastic flows with microstructure defects. Recently Riemannian asymmetric stresses in magnetohydrodynamics (MHD) have been considered by Billig (2004). **PACS numbers:**

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I Introduction

Recently two papers on the use of stresses magnetic[1, 2] or nonmagnetic [3] have been considered by Billig [1], Bonnano and Urpin [2] and L'ov et al [3]. In the first the Riemannian geometry was used along asymmetric stresses in MHD equations. In the second, Bonnano et al showed that the presence of magnetic field could induce instabilities in Couette plane flows , and in the third also instabilities in polymer interactions of fluids were investigated also in MHD. All these three papers together motivated us to use Elie Cartan idea [1] of associating a non-Riemannian asymmetric connection known nowadays by the name of Cartan torsion tensor [5, 6, 7], to moment or torque stresses to a Couette plane flow which has been systemmatically used over the years to investigate its Lagrangean stability. Cartan torsion has been also used in the fields of gravitation and cosmology, where torsion is in general associated with a spin density, often called Einstein-Cartan (EC) theory [8], or in modern language of theoretical physics, a torsioned high-energy menbranes [9]. Analogies between torsion in defects and in solids [10] and EC gravity has been put foward by Maugin [11] and Kröner [12] and developed most recently by Epstein [13]. Despite Cartan first motivation to apply torsion to physics was Einstein general relativistic cosmology, he argued that this study could be done easily by investigating the mechanical torque stresses [4] in three-dimensional, which could be exactly due to asymmetries in coupled shear stresses. In this brief report, we give an example that torsion can be applied also in physics of fluids, when the fluid, in the case a Couette type flow, is viscous and sheared. Actually Kambe [14] has previously investigate Couette planar flows stability, and showed that Riemann curvature tensor would exist in the case of existence of pressure, even a constant one. The stability of the Couette planar flow is also obtained by him, using the symmetry in the case of symmetric scalar stress potential and by making use of the technique of Ricci sectional curvature [15], where the negative sectional curvature indicates instability of the flow, in the Lagrangean sense. This idea can be easily shifted to flows in the case they have strong shear and viscosity and are able to support these antisymmetric stresses. In this brief report, we also show that allowing the presence of shearing stresses, and associating torsion to these stresses, the Couette flows are can be stable even in the Lagrangean sense. This can be explained by the fact, that as we

shall show, velocities involved in the fluid are inversely proportional to the flow torsion, and as torsion grows the velocity in the fluid decreases, which is a damping like physical effect in the flow. Actually, Kondo [10] has shown that the antisymmetry of stress tensor derivatative is associated with Cartan torsion. To test the range of this Lagrangean stability, we assume a marginal stability and put the sectional curvature for the torsioned connection to zero, and find an algebraic equation to previously arbitrary velocities in the fluid. Torsion would only allows for the instability if its sectional curvature is negative. A previously application of a non-torsion free connection in fluids, in the context of solitons, have been investigated by Ricca [16]. Rakotomanana [17] in turn have investigated NR manifolds with Cartan torsion, in the context of the geometrical approach to the thermodynamics of dissipating continua. The non-Riemannian geometry of vortex acoustic flows have been also recently addressed [18]. The paper is organized as follows: Section II presents a brief review of non-Riemannian geometry in the coordinate free language. Section III presents model, along with computation of the Lagrange stability of non-Riemannian Couette flow. Section IV presents the conclusions.

II Sectional non-Riemannian curvature

In this section, before we add we make a brief review of the differential geometry of surfaces in coordinate-free language. The Riemann curvature is defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{II.1}$$

where $X \epsilon T\mathcal{M}$ is the vector representation which is defined on the tangent space $T\mathcal{M}$ to the manifold $\mathcal{M}$. Here $\nabla_X Y$ represents the covariant derivative given by

$$\nabla_X Y = (X.\nabla)Y \tag{II.2}$$

which for the physicists is intuitive, since we are saying that we are performing derivative along the X direction. The expression $[X,Y]$ represents the commutator, which on a vector basis frame $\vec{e}_l$ in this tangent sub-manifold defined by

$$X = X_k \vec{e}_k \tag{II.3}$$

or in the dual basis ∂_k

$$X = X^k \partial_k \tag{II.4}$$

can be expressed as

$$[X, Y] = (X, Y)^k \partial_k \tag{II.5}$$

In this same coordinate basis now we are able to write the curvature expression (II.1) as

$$R(X, Y)Z := [R^l{}_{jkp} Z^j X^k Y^p] \partial_l \tag{II.6}$$

where the Einstein summation convention of tensor calculus is used. The expression $R(X, Y)Y$ which we shall compute bellow is called Ricci curvature. The sectional curvature which is very useful in future computations is defined by

$$K^{Riem}(X, Y) := \frac{< R(X, Y)Y, X >}{S(X, Y)} \tag{II.7}$$

where $S(X, Y)$ is defined by

$$S(X, Y) := ||X||^2 ||Y||^2 - < X, Y >^2 \tag{II.8}$$

where the symbol $<, >$ implies internal product. In the non-Riemannian (NR) case, the torsion two-form $T(X, Y)$ is defined by

$$T(X, Y) := \frac{1}{2}[\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]] \tag{II.9}$$

where $\bar{\nabla}$ is the non-Riemannian connection [7] endowed with torsion. As in EC theory [5] the geodesic equation does not depend on torsion; only Jacobi deviation equation depends on torsion which is enough for investigate the role of torsion on stability. Since the Jacobi equation is given by

$$\frac{d^2 J}{ds^2} = [||\nabla_{\vec{t}} \vec{e}_J||^2 - K^{NR}(t, \vec{e}_J)||J|| \tag{II.10}$$

where $||\vec{e}_J|| = 1$ and J is the Jacobi field, representing the separation between geodesics,while $\vec{t}$ is the geodesic tangent vector. Here $K^{NR}(X, Y)$ is given by

$$K^{NR}(X, Y) = K^{Riem}(X, Y) + 2 < T(X, Y), \bar{\nabla}_Y X > \tag{II.11}$$

Here as we shall see bellow the geodesic equation is $\nabla_Y Y = 0$ simplified this expression. Note from this expression that the instability, or separation of the geodesics in the flow

$$\frac{d^2 J}{ds^2} \geq 0 \tag{II.12}$$

implies that $K^{Riem} < 0$ which is the condition for Lagrange instabillity.

III Couette shear flow stability in non-Riemannian background

In this section we shall consider the Couette constant pressure, planar shear flow [11]

$$Y = (U(y), 0, 0) \tag{III.13}$$

with constant pressure p where $U(y) = y$ and X is given by

$$X = \vec{v}_l e_l \tag{III.14}$$

where $e_l := exp[i(\vec{l}\vec{x})]$ and $\vec{x} = (x, y, z)$ and $\vec{l} := (l_x, l_y, l_z)$ is the wave number. Here $\vec{v}_l$ represents an arbitrary velocity, which shall be determined below in order to generate non-Riemannian stability of Couette shear flows. The hydrodynamics in $\mathcal{R}^3$ Euclidean space [12] is given by

$$\bar{\nabla}_X Y = (X.\nabla)Y + grad p_{XY} \tag{III.15}$$

where the covariant derivative on the RHS of this equation [12] is

$$(X.\nabla)Y = e_l(\vec{v}_l.\nabla)(U(y), 0, 0) = e_l({v^y}_l U'(y), 0, 0) \tag{III.16}$$

A simple computation led Kambe to the result

$$\nabla^2 p_{XY} = \nabla^2 p_{YX} = -il_x {v_l}^y e_l \tag{III.17}$$

since $U'(y) = 1$. Note however that the equation (III.17) does not necessarily implies that $p_{XY} = p_{YX}$, since this is a sufficient but not necessary solution in the mathematical language. This can be easily seen by the argument that the equation (III.15) is equivalent to

$$\nabla^2[p_{XY} - p_{YX}] = 0 \tag{III.18}$$

and since $\nabla^2 = \nabla.\nabla$ we have that

$$\nabla p_{XY} - \nabla p_{YX} = \vec{c} \tag{III.19}$$

where $\vec{c}$ is an arbitrary constant. As we shall show this reasoning leads exactly to Cartan torsion 2-form $T(X, Y)$ which using the expression

$$p_{XY} = -il_x {v_l}^y \tag{III.20}$$

implies that Cartan torsion vector can be expressed as

$$T(X,Y) = grad(p_{XY} - p_{YX}) = [T^k{}_{21} X^2 Y^1]\partial_k \tag{III.21}$$

where the covariant components of torsion are

$$T^k{}_{21} = \frac{c}{{v_x}^l U(y)} = \frac{c}{{v_x}^l y} \tag{III.22}$$

An example of totally skew-torsion [19] has shown also be presented in the cholesteric blue phase of liquid crystals. The torsion vector (III.22) in the NR sectional curvature obtains

$$2 < T(X,Y), \bar{\nabla}_Y X >= (2\pi)^3 U'^2 \frac{{m^2}_x}{m^2} |{v_y}^m|^2 \tag{III.23}$$

which comes from the relation

$$2 < T(X,Y), \bar{\nabla}_Y X >= K^{Riem}(X,Y)(2\pi)^3 U'^2 \frac{{m^2}_x}{m^2} |{v_y}^m|^2 \tag{III.24}$$

which is the condition for marginal stability of

$$K^{NR}(X,Y) = 0 \tag{III.25}$$

Making use of the expression for the covariant derivative in the Riemann-Cartan connection $\bar{\nabla}$

$$\bar{\nabla}_Y X = i m_x e_m y \vec{v_m} - \frac{m_x}{m^2} e_m \vec{m} \tag{III.26}$$

Substitution of the value of torsion form above and this covariant derivative into expression (III.23) yields

$$i m_x U(y) e_m U(y) < \vec{v_m}, \vec{c} > - \frac{m_x}{m^2} {v_m}^y U'(y) e_m < \vec{m}, \vec{c} >= (2\pi)^3 U'^2 \frac{{m^2}_x}{m^2} |{v_y}^m|^2 \tag{III.27}$$

Thus since $U(y) = y$ this equation is simply solved when we choose a direction for $\vec{v_m}$. The simplest choice is $\vec{c} = c\vec{m}$, where $< \vec{m}, \vec{m} >= m^2$. Since [12] $< v_m, \vec{m} >= 0$, which upon substitution into (III.27) yields

$${v_m}^y = - \frac{m^2 c}{(2\pi)^3} \tag{III.28}$$

Physically this means that the curvature acts as a damping to the flow. Substitution of the value of the torsion scalar c into the value of torsion component one obtains

$$T^k{}_{21} = - \frac{1}{2} \frac{(2\pi)^3 {v^y}_m}{m^2 {v^l}_x y} \tag{III.29}$$

A simple observation of this expression shows that the torsion has the proper units of $\sim L^{-1}$ where L represents the length scale in the flow or liquid crystal.

IV Conclusions

One of the most important features of the investigation of the stability of flows in the Euclidean manifold $\mathcal{E}^3$, in some detail the stability of an incompressible or volume preserving flow, using the method of the sign of the Ricci sectional curvature. Also important is the issue of stability in plasma astrophysics as well as in fluid mechanics. In this paper we discuss and present the contribution of Cartan torsion tensor on damping the Riemannian Lagrange instabilities of viscous Couette shear flows. We could say that contrary to the Riemannian case [12] where the flow is unstable in the Lagrangean particle sense, but is neutrally stable, here the Couette shear flow may be fully stable for an appropriate chice of velocities. The role of torsion in crystal curvature frustration [20] could also be investigate concerning the stability of geodesics.

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